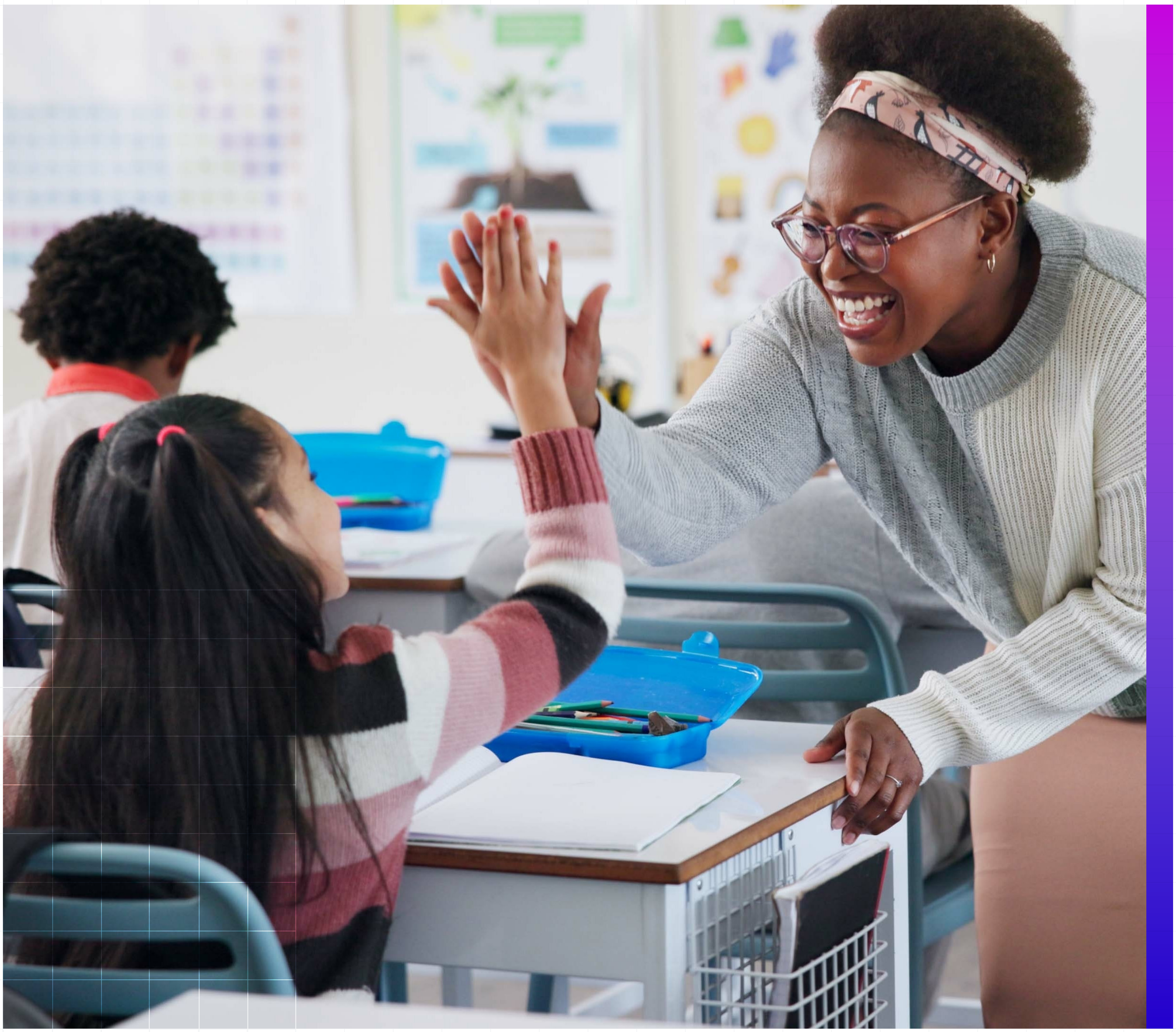
# Foundations of Numeracy



PowerMyLearning.org/Framework

# Introduction and Acknowledgements

## What does it take for every child to enter middle school with a strong foundation in mathematics?

**This question has echoed across classrooms, district offices, and statehouses nationwide.** While literacy has benefited from decades of research and an emerging shared vision, math has lacked an equivalent anchor. As urgency grows to improve math outcomes, the field has lacked a common, evidence-informed framework defining what numeracy means in the elementary grades.

The Foundations of Numeracy was created to address this need. To ensure the framework reflects both the depth of the research and the realities of practice, PowerMyLearning partnered with leading experts to ground the Foundations of Numeracy in a rigorous, multi-disciplinary evidence base:

- **Extensive research synthesis**: Builds on the National Math Advisory Panel's 2008 work and integrates more than 200 studies across math learning science, developmental psychology, and mathematics education.
- **High standards of evidence**: Prioritizes research aligned with ESSA and What Works Clearinghouse criteria, including randomized controlled trials and meta-analyses.
- **Breadth of expertise**: Informed by leading scholarship on mathematical reasoning, fluency, and motivation.
- **Practice-based input**: Shaped by feedback from educators and district partners to ensure usability across classrooms and systems.

At its core, the Foundations of Numeracy provides a shared language for the essential building blocks of PK–5 learning. It helps policymakers, curriculum developers, education leaders, teachers, and parents/caregivers align efforts, ensuring coherence.

 Our hope is for it to become a trusted reference point, accelerating our collective focus on improving math outcomes for all students.

### About PowerMyLearning

PowerMyLearning is a national nonprofit dedicated to improving math outcomes for PK–5 students. For over 25 years, we've partnered with schools and districts—especially those serving multilingual learners and historically underserved communities—with evidence-based programs, services, and digital solutions proven to boost math achievement.

### Acknowledgements

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# Contents

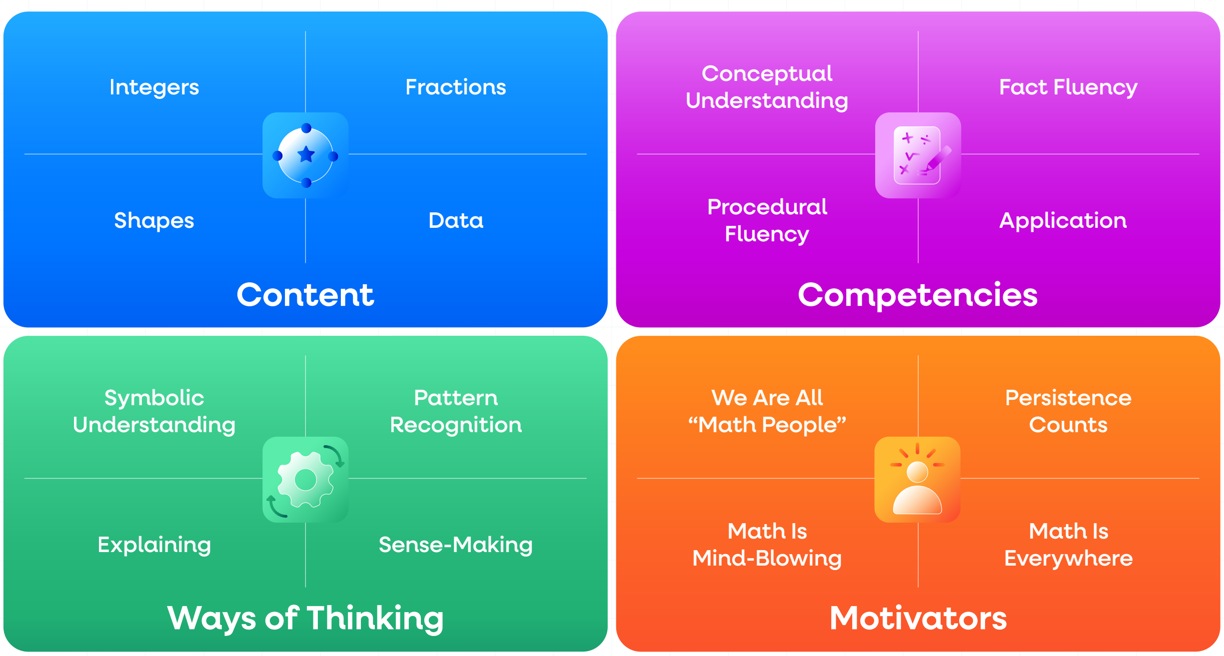
# Executive Summary

**Nationwide, momentum is building to strengthen math learning as proficiency gaps grow.**

Yet the sector continues to operate without the kind of shared, evidence-based guidance that helped transform literacy through the Science of Reading. While literacy has benefited from decades of sustained investment, expanding instructional time, systemwide supports, and training, math has not kept pace.

The stakes are high: math is a gateway to long-term opportunity. In fact, math proficiency predicts later academic achievement more strongly than reading, and students who develop mathematical conceptual understanding, math fluency, and math reasoning skills by fifth grade are better prepared for long-term success. But getting there requires clarity, including a common, evidence-informed reference point for the numeracy skills students in grades preK–5 need.

The Foundations of Numeracy provides that clarity. Synthesizing decades of research and expert insight, the framework identifies the essential building blocks of math learning and provides a common language to support coherent, evidence-driven decision-making. It enables system leaders, practitioners, and parents/caregivers to align around what students need most and ensure instruction is grounded in the practices that drive measurable impact.



# Executive Summary (continued)

The Foundations of Numeracy is a framework for understanding the evidence-backed requirements of elementary mathematics. PowerMyLearning, a national education nonprofit, has developed this numeracy framework based on an examination of the research on pre-K–5 math learning from the past three decades (pre-K, in this context, refers to formal classroom-based programs for 4- and 5-year olds). It attempts to build on the previous and essential work by the National Math Advisory Panel from 2008 and integrate more recent work (including that of Common Core authors) into a simpler and easier-to-recall structure. It is focused on "what" numeracy is, not "how" to teach it. This framework can be used by educators in their own lesson planning and instructional decision-making about students' needs, by school and system leaders evaluating investments in instructional materials (both core and supplemental) and professional learning, and advocates and policy-makers interested in stronger early math outcomes.

The Foundations of Numeracy includes four ***"Cornerstones:"*** **Content, Competencies, Ways of Thinking,** and **Motivators**. Each Cornerstone includes four individual Building Blocks, resulting in 16 total Building Blocks for a strong Foundation of Numeracy by the end of fifth grade. The four Content areas critical for elementary math students are: Integers, Fractions, Shapes, and Data.

**The four Competencies are:**
**1: Conceptual Understanding**
**2: Fact Fluency**
**3: Procedural Fluency**
**4: Application**

These are mutually-reinforcing and all four are essential. The four Ways of Thinking—Symbolic Understanding, Pattern Recognition, Explaining, and Sense-Making—together form the foundation of critical thinking.

The four Motivators are productive beliefs about math which help students feel motivated to learn: We Are All "Math People"; Persistence Counts; Math Is Mind-Blowing!; and Math Is Everywhere. Together, the Foundations of Numeracy equip students with what they need to learn pre-algebra, algebra, and higher levels of math and ultimately, for productive and joyful engagement in all the ways math is used in the world.

Although all 16 of the distinct Building Blocks are essential and evidence-backed parts of a robust and stable numeracy foundation, educators should recognize that some Building Blocks require more focused instructional time than others. Specifically, the Competencies and Content must receive adequate instructional attention. The Ways of Thinking can often be developed through the process of instruction in the skill progressions inherent in the Content and Competencies Cornerstones, while the Motivators can be effectively instilled indirectly or integrated in a teacher's attitude and approach to instructional activities related to other Building Blocks. Knowing what each student needs to build a strong numeracy foundation is at the heart of being an effective math teacher.

This document provides a brief overview of each Building Block, explains why it is an essential part of the Foundations of Numeracy, describes ways that teachers can support student learning, and provides examples of evidence-based, grade-appropriate learning activities.

## Cornerstone 1: Content

# Everything Builds From These

These content topics undergird all future math study.

### Integers
Know symbols for quantities, place value, the four operations, and estimation. Includes positive and negative whole numbers and zero.

### Fractions
Use rational numbers to represent parts and proportional relationships including fractions, decimals, percentages, and ratios; applications in money, time, and distance.

### Shapes
Describe and measure the physical world using geometric and quantitative characteristics, including 2D and 3D figures.

### Data
Represent, interpret, and communicate numerical data visually and with words.

**Numeracy requires mastery of four discrete but interconnected topics.**

The four Building Blocks within the Content Cornerstone are: Integers, Fractions, Shapes and Data.

Together, proficiency with these topics will equip learners with the foundational skills necessary to understand the world quantitatively and solve problems requiring mathematical reasoning of all types.

## Content: Everything Builds From These

# Integers

**Integers**
Know symbols for quantities, place value, the four operations, and estimation. Includes positive and negative whole numbers and zero.

**Fractions**
Use rational numbers to represent parts and proportional relationships including fractions, decimals, percentages, and ratios; applications in money, time, and distance.

**Shapes**
Describe and measure the physical world using geometric and quantitative characteristics, including 2D and 3D figures.

**Data**
Represent, interpret, and communicate numerical data visually and with words.

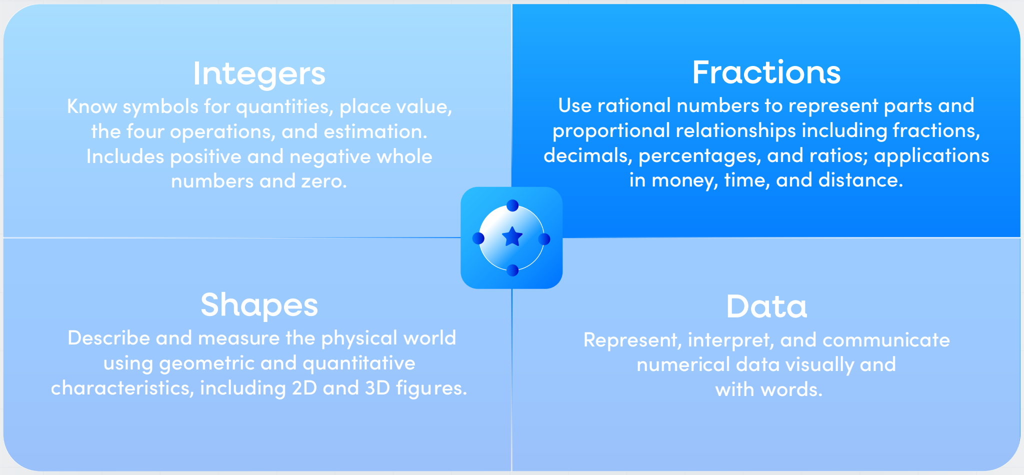
Integers are the basis for everything students will learn later, including arithmetic operations, algebra and geometry. Students begin with whole numbers, including zero, growing in quantity over time, and then encounter real-world contexts involving negative whole numbers in later grades. Mastering Integers includes knowing the symbols that represent quantities, understanding place value, developing the ability to compose and decompose numbers, performing basic operations accurately and fluently (addition and subtraction by third grade and multiplication and division by fifth grade), and estimating when appropriate.

Examples:

- **Pre-K:** Students count up to five objects and match them to a numeral card.
- **Grades K–2 :** Students listen to and tell each other simple story problems, such as "Tom has five toy cars. He gives two to his friend. How many toy cars does Tom have left?" They use physical objects, draw pictures, and then write a number sentence that corresponds with the story.
- **Grades 3–5:** Students use expanded notation to decompose a three-digit number, multiply it by a single-digit number, and create a graphical representation to visualize this mathematical process.

## Content: Everything Builds From These

# Fractions




Also essential to numeracy is the ability to understand and use all the forms of rational numbers that represent parts and proportions, specifically: fractions, decimals, percentages and ratios. Representing, converting, and manipulating these forms comfortably is pivotal to students' numerical development and fundamental to higher-order math learning and reasoning—from algebra to calculus. Students must be able to execute the four operations (i.e., fractions, decimals, percentages and ratios) and understand equivalence across the forms. Instructional examples should extend beyond the classic but simplistic slices of pizza to include a wide variety of applications, including in time, measurement, finance, statistics, engineering, and daily life. The number line, in particular, is an important tool for deepening students' understanding of fractional parts. Analyses of large, nationally representative longitudinal datasets from the United States and the United Kingdom revealed that elementary students' knowledge about fractions uniquely predicts their algebra knowledge and overall high school math achievement, even after statistically controlling for other types of math knowledge, general intellectual ability, working memory, and family income, and education.

Examples:

- **Pre-K:** Students put together simple shape puzzles made of two or three large pieces to form a whole shape.
- **Grades K–2 :** A teacher uses a 0–1 paper number line to introduce the idea that a whole can be divided into half and then into quarters; students fold and mark their paper number lines and color each quarter.
- **Grades 3–5:** Students keep a diary that tracks their weekend time and then add the fractional hours to calculate the total number of hours they spend on their favorite activities.

## Content: Everything Builds From These

# Shapes

### Integers
Know symbols for quantities, place value, the four operations, and estimation. Includes positive and negative whole numbers and zero.

### Fractions
Use rational numbers to represent parts and proportional relationships including fractions, decimals, percentages, and ratios; applications in money, time, and distance.

### Shapes
Describe and measure the physical world using geometric and quantitative characteristics, including 2D and 3D figures.

### Data
Represent, interpret, and communicate numerical data visually and with words.

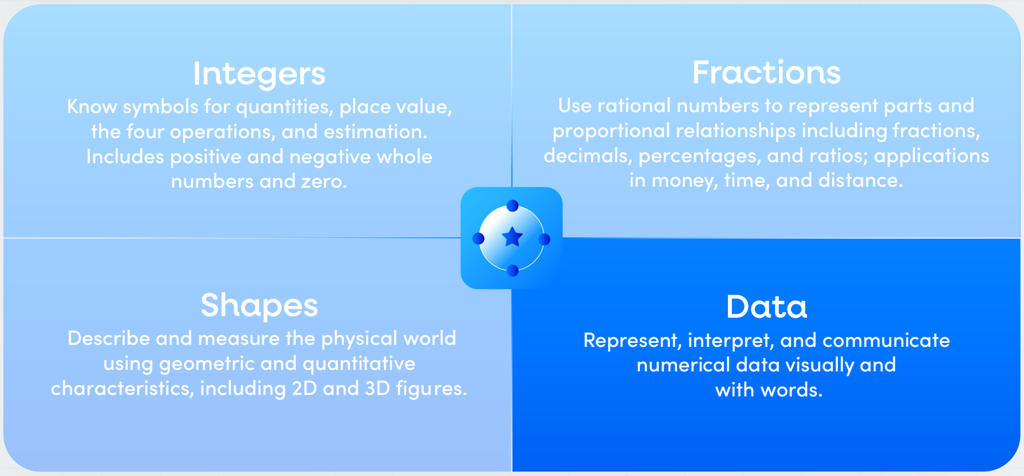
Students should be able to understand and describe the physical world using geometric characteristics (e.g., shape, orientation), quantitative measurements and vocabulary. A strong grasp of two-dimensional and three-dimensional figures and measurement helps students transition from concrete visual understanding to abstract mathematical thinking. For example, understanding the properties of shapes can lead to recognizing algebraic patterns, and knowing how to measure accurately prepares students for solving equations that involve geometric concepts like area and perimeter. Mastery of shapes and measurement in elementary school develops spatial reasoning which sets the stage for students to excel in more advanced math and later success in STEM fields.

Examples:

- **Pre-K:** Students go on a simple shape hunt, finding objects with familiar shapes and naming the basic shape they see.
- **Grades K–2 :** Students use different tools to measure dimensions of different objects in the classroom using standard units and comparing the relative dimensions of the objects.
- **Grades 3–5:** Students use grid paper to place playground equipment on a playground, using different shapes for each type of equipment, with a limit of 100 square meters. They calculate the area of the footprint of each piece of equipment to ensure it fits into the available space, compare designs, and identify patterns.

## Content: Everything Builds From These

# Data




Elementary students should be able to represent, interpret and communicate numerical data visually and with words. This lays the foundation for more complex statistical thinking later and supports future success with real-world, "21st century" skills such as navigating conflicting information, thinking critically about evidence, and making informed, context-based decisions. Being proficient with data in the elementary grades includes being able to read and interpret simple data displays such as tables, lists, and basic graphs and create understandable data visualizations by counting, sorting and organizing data. Counterintuitively for most elementary grade teachers, the simple monochromatic bar, column, and pie charts—rather than pictographs—are best for supporting students' conceptual understanding of these data visualization tools.

Examples:

- **Pre-K:** Students sort 6–8 images into two groups (e.g. cats and dogs) and count the number of images in each group.
- **Grades K–2 :** Students design and administer surveys about class preferences (e.g., sports, food), create and present data visualizations, and discuss which visualizations were most helpful and why.
- **Grades 3–5:** Students collect data about the weather each morning, represent the survey results graphically and analyze trends in different months.

## Cornerstone 2: Competencies

# All Four Are Mutually Reinforcing

They are mutually reinforcing. Students must work on all four.

### Conceptual Understanding
Demonstrate number sense; comprehend the principles behind mathematical concepts.

### Fact Fluency
Use math facts with confidence and fluidity.

### Procedural Fluency
Execute operations accurately and efficiently.

### Application
Use math in real-world situations, such as money, time, and distance.

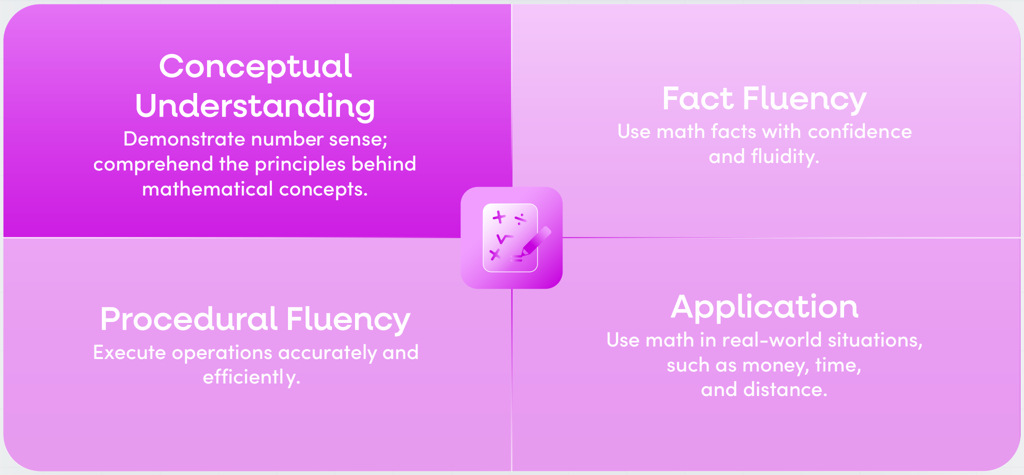
**The Competencies describe what elementary students are able to do mathematically.**

The four Building Blocks within this Cornerstone have often been pitted against each other, based on a mistaken belief that they are mutually exclusive. But this creates a false dichotomy; the “math wars” are an unfortunate distraction.

Conceptual understanding and computational fluency are not mutually exclusive aims. In fact, the evidence is clear that fluency with facts and procedures helps students with conceptual understanding and vice versa. Numeracy requires fluency with facts and procedures as well as conceptual understanding and the ability to apply these mathematical capabilities to situations in the real world.

## Competencies: All Four Are Mutually Reinforcing

# Conceptual Understanding



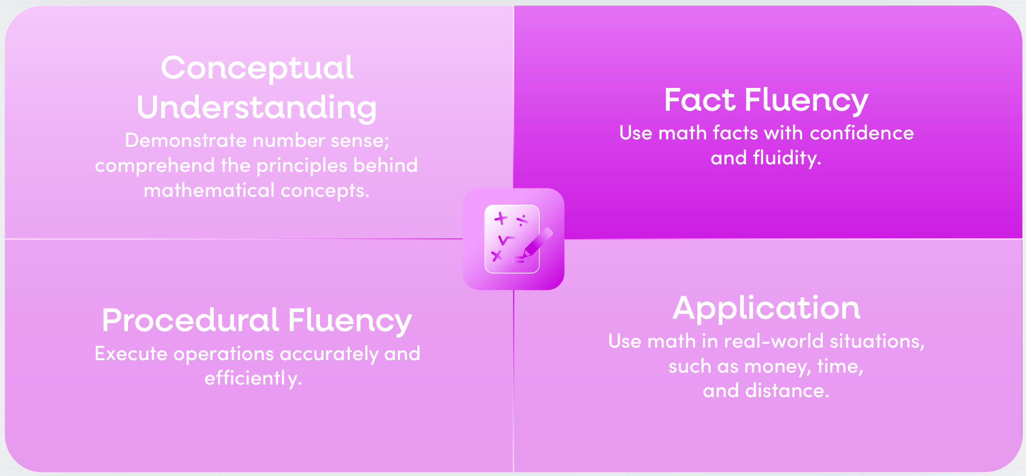

Students must be able to grasp the underlying ideas, relationships, and principles behind mathematical concepts. Developing "number sense" in the earliest years is a non-negotiable component of numeracy; number sense is the intuitive understanding of numbers and quantities as well as their relationships, patterns, and uses in various contexts. Later, students should know why a mathematical strategy, procedure, or algorithm works and how the idea relates to other mathematical ideas, including ideas they already know. Conceptual understanding supports retention and the transfer of learning to new problems. For example, when students understand that fractions represent parts of a whole, they can apply this knowledge to different areas of math such as geometry and measurement, producing flexible mathematical thinking. For many students, a helpful learning sequence for developing Conceptual Understanding starts with experiencing the mathematical idea in concrete form first, then representing the idea in pictures, and finally communicating the idea abstractly using numbers and symbols (i.e., Concrete-Representational-Abstract, or "CRA"). Some foundational concepts that learners should understand by the end of elementary school include: that numbers represent quantities; place value; equivalence; commutative, associative and distributive properties; proportional reasoning; units of measurement; symmetry and congruence; and probability.

Examples:

- **Pre-K:** Students use a small set of objects to make two equal groups, practicing matching one object to one object to build their understanding of one-to-one correspondence.
- **Grades K–2 :** Students use counting tiles, simple drawings, and number bond diagrams to represent that $8 + 7 = (5 + 3) + 7 = 5 + (3 + 7) = 5 + 10 = 15$ to build and demonstrate their Conceptual Understanding of equivalence, place value, and decomposition.
- **Grades 3–5:** Students predict and record the outcomes of games in a carnival (e.g., coin toss, dice roll, card draw, spinner) to develop and demonstrate their Conceptual Understanding of certain, likely, unlikely and impossible outcomes as part of a foundation for learning probability.

## Competencies: All Four Are Mutually Reinforcing

# Fact Fluency




Fact fluency is the comfortable and fluid use of addition, subtraction, multiplication, and division facts. (These math facts are sometimes defined as 390 basic math facts: 0 + 0 to 9 + 9 for addition and their corresponding subtractive inverses; and 0 x 0 to 9 x 9 for multiplication and their corresponding division inverses, with the exception of division by zero). Students develop memories of math facts, by learning and using effective strategies and encountering math facts in different ways. Fact Fluency frees up cognitive resources (e.g., working memory) for multi-step problem-solving tasks and understanding more complex mathematical concepts. For example, a sixth grade student asked to solve an algebraic equation like 3x + 7 = 16 who can quickly recall that 16 – 7 = 9 and 3 x 3 = 9, can focus on the idea of solving for an unknown number rather than getting bogged down in basic calculations. Another student may see that something (3x) plus 7 is 16 and realize the 3x represents 9, using math facts to find that x is 3. Students should work on increasing their fluency after developing consistent accuracy, and daily practice is recommended to achieve automaticity. By second grade, students should have fluency with addition and subtraction of whole numbers up to 20; by fifth grade they should master multiplication and division of whole numbers. Note: memorization and retrieval practice doesn't need to be boring, tedious or anxiety-provoking.

Examples:

- **Pre-K:** Students use counters or dot cards to quickly recognize small quantities (1–3) shown in different arrangements without counting.
- **Grades K–2 :** Students play *Combinations of 10 Bingo*. When the teacher calls a number, students find its "partner to 10" on their board (e.g., if the teacher says three, students look for a "seven" on their Bingo board and mark it out).
- **Grades 3–5:** Students who know their fives multiplication combinations practice their sixes by rolling a 10-sided die. They reason: "five groups of [my number] equals _, so six groups of [my number] equals _," using known facts to build new ones.

## Competencies: All Four Are Mutually Reinforcing

# Procedural Fluency

**Conceptual Understanding**
Demonstrate number sense; comprehend the principles behind mathematical concepts.

**Fact Fluency**
Use math facts with confidence and fluidity.

**Procedural Fluency**
Execute operations accurately and efficiently.

**Application**
Use math in real-world situations, such as money, time, and distance.

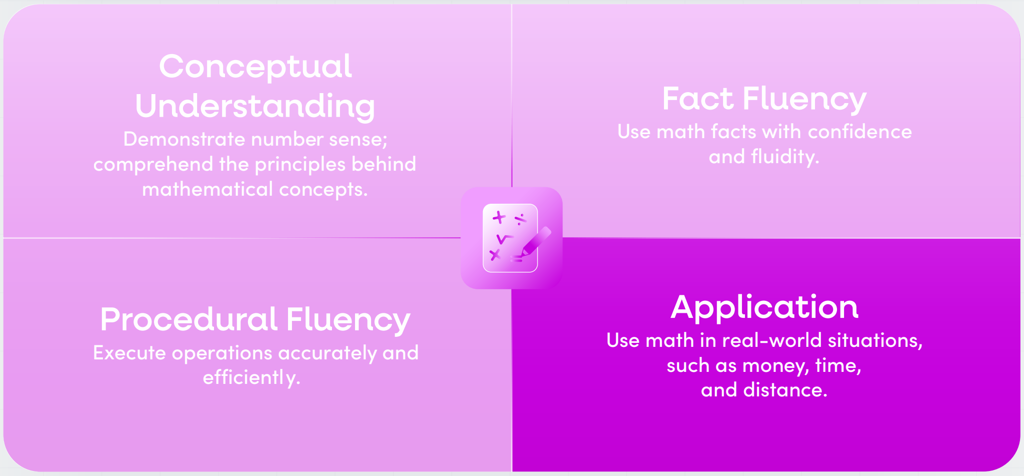
Students should be able to execute mathematical operations flexibly, accurately and efficiently. Similar to Fact Fluency, fluency with generalizable procedures reduces the cognitive load and enhances students' ability to engage in more complex problem-solving tasks by allowing them to focus on the underlying mathematical concepts. Furthermore, knowing how to properly carry out arithmetic operations builds confidence in math, and provides the structure and reliability needed to tackle more advanced math. For example, students should be able to carry out addition and subtraction with whole numbers up to 100 by the end of second grade, and multiplication and division with fractions by the end of fifth grade.

Examples:

- **Pre-K:** Students count a small group of up to five objects using one-to-one correspondence and practice recounting the same group after the objects are moved or rearranged to keep their counting accurate.
- **Grades K–2 :** A student asked to subtract 25 from 44 notices that this problem is very similar to 45 – 25, which is 20 and adjusts their answer to 19 (since they added one to start the problem). Another student jumps up, thinking 25 to 35 is 10 and 35 to 44 is 9 more, correctly providing 19 as the answer.
- **Grades 3–5:** Students adding 1/4 and 1/3 find a common denominator (12), convert each fraction to an equivalent fraction with the common denominator (3/12 and 4/12), add the numerators (7), and write the resulting fraction with the common denominator (7/12).

# Competencies: All Four Are Mutually Reinforcing

# Application




Students should experience the practical use of mathematical concepts as early as possible, in ways that are relevant to their experiences. They should learn to understand and describe a situation in mathematical terms and think about different ways to find solutions. This helps learners build confidence and flexibility in their mathematical thinking, fostering a deeper understanding of mathematical concepts and setting the stage for more complex reasoning in later grades. For example, students who have learned to multiply fractions can strengthen their understanding by using it with a recipe one day and calculating distance traveled over multiple trips the next day. Currency, time, distance, temperature, volume, and types are all useful and relevant applications for the concepts and operations essential for numeracy.

Examples:

- **Pre-K:** Pairs of students play a ring toss game using two colors of rings. Each student tosses a small set of up to 5 rings. After tossing, they count how many rings of each color landed on the target and decide which color won the game.
- **Grades K–2 :** Students make snack mixes containing different quantities of each ingredient and write recipe cards (e.g., one cup raisins, one cup pretzels, two cups M&Ms).
- **Grades 3–5:** Students decide what to buy for a class party with a limited budget. They research the cost of supplies, decide what quantities to buy for each item, and ensure that they do not exceed their budget.

## Cornerstone 3: Ways of Thinking

# Practice Early To Develop Critical Thinking

Practicing these early yields critical thinking later.

### Symbolic Understanding

Represent quantities and mathematical ideas using symbols and diagrams.

### Pattern Recognition

Notice and use patterns to understand mathematical structure, generalize, and act accordingly.

### Explaining

Articulate mathematical and numerical reasoning in words.

### Sense-Making

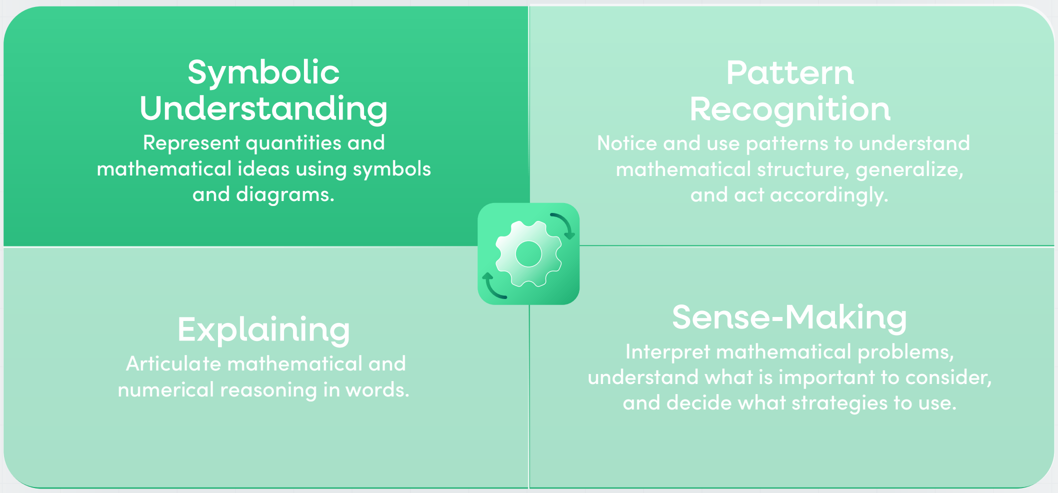
Interpret mathematical problems, understand what is important to consider, and decide what strategies to use.

**These four cognitive approaches to interacting with ideas about the world are the beginnings of critical thinking.**

They can and should be developed through practice starting at an early age. Note: The “Practices” described in the Common Core State Standards for math are distilled into these Ways of Thinking.

## Ways of Thinking: Practice Early To Develop Critical Thinking

# Symbolic Understanding



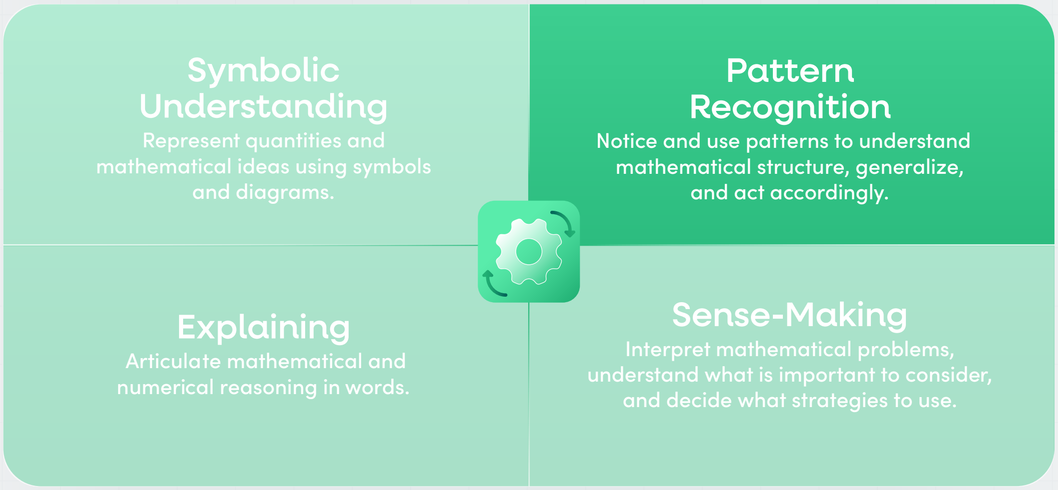

Moving beyond concrete experiences and being able to represent quantities and mathematical ideas using symbols is one of the most crucial abilities for more advanced mathematical thinking. It enables students to interpret symbols in a mathematical context, translate word problems into numbers and communicate their reasoning in generalized ways. For example, when a student understands that the number five represents five apples, toys, or any set of five items, they can begin adding and subtracting without needing to see or touch the objects. Teachers can help students develop this habit through an instructional routine that includes providing a concrete experience followed by a visual representation, and then explicitly making a connection to a mathematical expression; this "CRA" pattern is one of the most evidence-backed pedagogical practices for math instruction.

Examples:

- **Pre-K:** Students look at simple dot cards showing 1–3 dots and use their fingers or tally marks to show the same quantity.
- **Grades K–2:** Students match pictures, stories, and number equations with arrangements of base-ten blocks.
- **Grades 3–5:** Students use fraction strips to compare and order fractions with different denominators by size relative to each other and to whole numbers. In reflection, they are asked to describe the meaning of the fraction bar.

## Ways of Thinking: Practice Early To Develop Critical Thinking

# Pattern Recognition



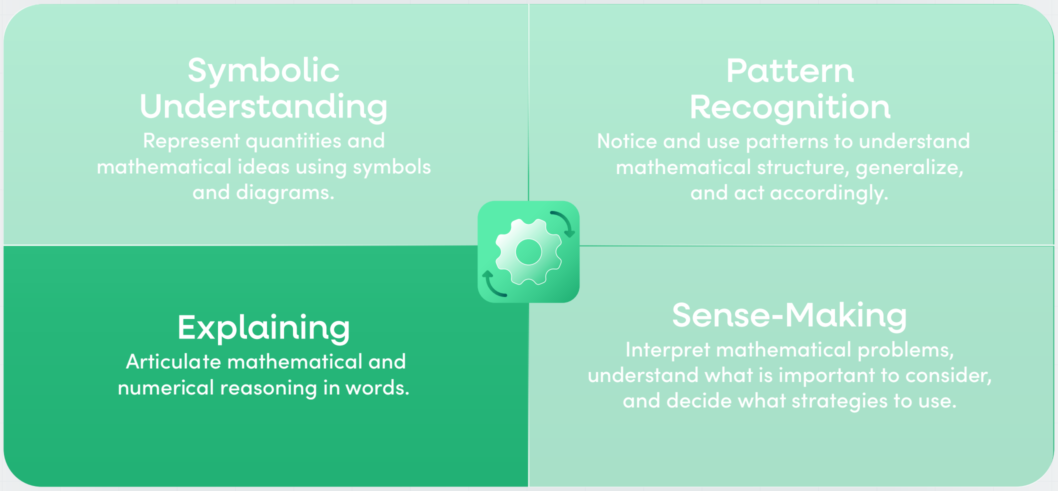

Elementary students developing numeracy must develop the ability to notice patterns in mathematical contexts to understand concepts more deeply and use those patterns to solve problems and strategically deploy mathematical tools. For example, noticing that two, four, six, eight is a pattern helps children predict the next number and is a precursor to understanding addition and multiplication. This habit of making connections and generalizing mathematical ideas leads to more flexible thinking and lays the groundwork for algebraic reasoning. Teachers can provide students with mathematical patterns to explore and describe.

Examples:

- **Pre-K:** A teacher creates a repeating sound pattern using claps, stomps, and taps. Students listen to the pattern and try to recreate it, then create their own simple sound pattern for classmates to copy.
- **Grades K–2:** Students are given a sequence of whole numbers decreasing by a consistent rate with missing numbers; they must identify the missing numbers in a sequence and explain the pattern.
- **Grades 3–5:** Students explore the pattern in a sequence of fractions with the same denominator (1/8, 2/8, 3/8) using fraction strips, graphing and words to describe the pattern.

# Ways of Thinking: Practice Early To Develop Critical Thinking

## Explaining



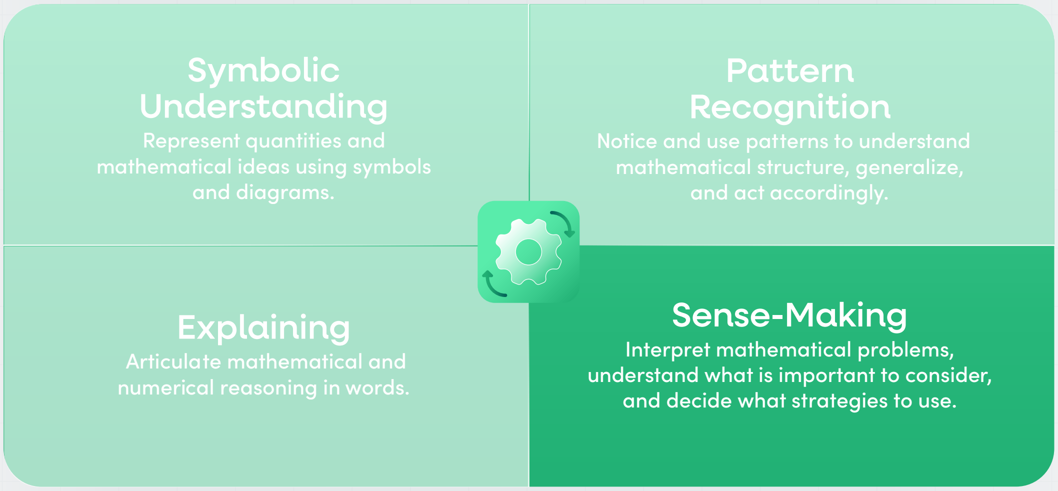

As students communicate their mathematical and numerical understanding in words to explain their thinking, articulate reasoning and justify solutions, they strengthen their neural connections, deepen their Conceptual Understanding and build the ability to transfer knowledge to new situations. Explaining also helps students clarify their thinking, and fix and learn from mistakes. Doing this requires creating an environment in which students feel comfortable sharing their ideas, providing clear instruction on math vocabulary to help students effectively communicate mathematical ideas, and ensuring students practice using a variety of tools (e.g., objects, drawings, and diagrams) to support their explanations.

Examples:

- **Pre-K:** As a whole class, students count small sets of up to five objects in two different piles. Students are asked to say which pile has more or fewer and explain how they know.
- **Grades K–2:** A teacher uses the math language routine, “Stronger and Clearer Each Time,” in which students explain how they got their answer to a math task to a partner, the partner asks a question (e.g., Why did you move these tiles?), and the explainer incorporates their answer to the question into a new, clearer explanation for a second partner.
- **Grades 3–5:** After students are asked to solve the division problem 364 ÷ 4, the teacher asks students who used different strategies to each present their approach.

## Ways of Thinking: Practice Early To Develop Critical Thinking

# Sense-Making




Sense-Making involves interpreting and making meaning from information, which enables students to understand mathematical problems and real-world situations. For elementary students, it involves analyzing and understanding a problem well enough to plan an approach and applying an appropriate set of strategies. In the earliest grades, this may entail deciding when to use physical resources like counting blocks, number lines, and drawings to visualize math concepts, while upper elementary students may decide when to use fractions instead of decimals to solve a problem. It also includes knowing when and how to estimate and when precision is important. This early ability to unpack mathematical situations lays groundwork for flexible thinking and solving more complex problems. To develop this habit, teachers should provide mathematical tasks and activities that encourage students to explore, use conjecture, and articulate their reasoning.

Examples:

- **Pre-K:** Students look at two containers of different sizes and predict which one will hold more blocks. They explain their thinking, then test their prediction using real blocks.
- **Grades K–2:** A teacher uses a story to convey 22 – 7 = 15 and asks students to use two different strategies for depicting this equation, including base-ten blocks, number line, drawings and decomposition. Students then explain which strategy they prefer.
- **Grades 3–5:** Students analyze a graph about the increase in the price of a gallon of milk over the past 10 years, answering questions about how much the price has gone up, what years the price increased the most, and what impact this has had on their household's budget.

## Cornerstone 4: Motivators

# Engagement Is Essential

What drives learning: engagement and believing in our potential.

### We Are All "Math People"

Believe in the innate ability of humans to understand mathematical ideas.

### Persistence Counts

Have a growth mindset and know that effort leads to learning.

### Math Is Mind-Blowing!

Appreciate the power and beauty of mathematics.

### Math Is Everywhere

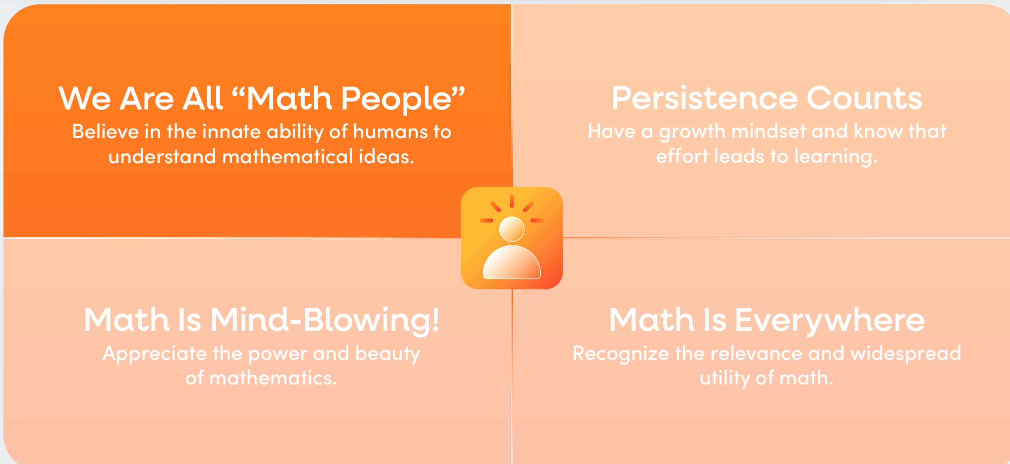
Recognize the relevance and widespread utility of math.

**In every field of human activity, a motivation to learn and a belief in one's own learning efficacy are essential prerequisites for learning and growth.**

Math is no different. Because modern American society contains many undermining narratives about math learning, attention to motivators is even more important, starting at the earliest ages.

## Motivators: Engagement Is Essential

# We Are All Math People




Contrary to the prevailing cultural narrative that some people are "math people" and others are not, every student has a mathematical mind that can be nurtured. In fact, studies have shown that humans have an intuitive grasp of quantity. Teachers, administrators and families must make intentional efforts to communicate that math is for everyone and everyone belongs in math. This requires explicitly promoting inclusive messages and countering negative ones, creating inclusive classroom environments, and establishing policies for support and acceleration rather than exclusivity.

Examples:

- **Pre-K:** Students explore a small group of objects arranged on the carpet and share what they notice or wonder about them. Children offer different ideas and the teacher highlights that everyone sees math in their own way.
- **Grades K–2:** Each month, a teacher showcases a famous mathematician, including women and people of color.
- **Grades 3–5:** A teacher uses a "pre-teaching" strategy, in which a student who has been struggling is tutored on the next day's lesson, so that student becomes the expert in the class on the new strategy.

## Motivators: Engagement Is Essential

# Persistence Counts


Students must believe in the power of effort to improve their math skills, and experience growth that comes from repeated and sustained effort. A growth mindset is important because it encourages students to embrace challenges, learn from mistakes, and persist in solving problems. Students with a growth mindset understand that struggling is a normal part of the learning process, so they are more likely to keep trying even when problems are challenging. This persistence builds confidence, encourages a love for learning, and leads to better long-term success in math. Students with a growth mindset attain higher academic performance compared to those with a fixed mindset. Teachers can cultivate this mindset by creating a supportive learning environment that values effort, provides constructive feedback, and encourages students to view mistakes as learning opportunities.

Examples:

- **Pre-K:** Students try to build a block tower as tall as a toy figure. When the tower falls or leans, they try again in a different way and talk about what helped them keep going.
- **Grades K–2:** Students track their own progress toward fluency with addition facts up to 20 and celebrate when they reach a goal they have set.
- **Grades 3–5:** Students use math journals to reflect on their problem-solving process, including challenges faced and how they overcame them, reinforcing the idea that effort leads to improvement.

## Motivators: Engagement Is Essential

# Math Is Mind-Blowing!

### We Are All "Math People"
Believe in the innate ability of humans to understand mathematical ideas.

### Persistence Counts
Have a growth mindset and know that effort leads to learning.

### Math Is Mind-Blowing!
Appreciate the power and beauty of mathematics.

### Math Is Everywhere
Recognize the relevance and widespread utility of math.

Students should experience the thrill that comes from understanding the power and beauty of mathematics. When students find math enjoyable—and even exciting!—they are more likely to actively participate in learning activities and develop a positive attitude toward the subject. Ideally, students recognize mathematics as a subject rich with interesting patterns, profound structures, and logical elegance. Curiosity about math is required for advanced mathematical thinking and problem-solving. Developing this mindset requires providing time and activities in which students can make connections, explore patterns, and share teachers' and classmates' enthusiasm for solving interesting math problems.

Examples:

- **Pre-K:** Students pour water between containers of different shapes and sizes and notice how the same amount of water can look tall, short, wide, or narrow. They share what surprised them.
- **Grades K–2:** Students look at artistic works (e.g., M.C. Escher, Salvador Dali, Islamic art, Native American weavings), and create their own art piece in that style, and explain some aspect of their art using geometric terms.
- **Grades 3–5:** Students are presented with various triangles and challenged to discover the relationships between the angles in the triangles, discovering that the sum of all angles in a triangle is always 180 degrees no matter the size or shape of the triangle.

## Motivators: Engagement Is Essential

# Math Is Everywhere

| | |
|---|---|
| **We Are All "Math People"**<br>Believe in the innate ability of humans to understand mathematical ideas. | **Persistence Counts**<br>Have a growth mindset and know that effort leads to learning. |
| **Math Is Mind-Blowing!**<br>Appreciate the power and beauty of mathematics. | **Math Is Everywhere**<br>Recognize the relevance and widespread utility of math. |

Math is part of kids' everyday lives through playing with blocks and puzzles, keeping score in sports, shopping, cooking, traveling, and telling time. Recognizing that math is all around them and not just confined to the classroom can increase motivation for learning by showing that math is a helpful tool for understanding and navigating the world. Teachers can help develop this mindset by connecting math to everyday situations.

Examples:

- **Pre-K:** Students help set the table by counting plates, cups, or napkins, noticing that they use math during everyday classroom routines.
- **Grades K–2:** Students ask family members for recipes and make a simple recipe in class.
- **Grades 3–5:** Students create a class poll that includes both categorical and numerical questions, create graphs, and discuss mean, median, and mode.


# Appendix

# Appendix 1

## Foundations of Numeracy - Summary

| | | |
|---|---|---|
| Content | Integers | Understand and use positive whole numbers, zero, and negative whole numbers. Includes place value, composing and decomposing numbers, performing all four operations, and estimating. |
| | Fractions | Understand and use all forms of rational numbers that represent parts & proportions, i.e.: fractions, decimals, percentages and ratios; execute four operations; apply broadly. |
| | Shapes | Understand and describe the physical world using geometric characteristics, quantitative measurements and vocabulary. Includes 2D and 3D. |
| | Data | Represent, interpret and communicate numerical data visually and with words. |
| Competencies | Conceptual Understanding | Demonstrate “number sense;” understand rationale and how ideas connect to other ideas. |
| | Fact Fluency | Confident and fluid recall of addition, subtraction, multiplication, and division facts (e.g., 390 basic math facts). |
| | Procedural Fluency | Execute mathematical operations accurately and efficiently. |
| | Application | Use mathematical concepts in real-world situations, such as money, time, and distance. |
| Ways of Thinking | Symbolic Understanding | Represent quantities and mathematical ideas using symbols. |
| | Pattern Recognition | Notice and use patterns to understand mathematical structure and generalize. |
| | Explaining | Translate mathematical and numerical understanding into words. |
| | Sense-Making | Interpret mathematical problems and decide what strategies to use. |
| Motivators | We Are All "Math People" | Believe in the innate ability of humans to understand quantity. |
| | Persistence Counts | Use a growth mindset and understand that effort leads to learning. |
| | Math Is Mind-Blowing! | Appreciate the power and beauty of mathematics. |
| | Math Is Everywhere | Recognize the relevance and widespread utility of math. |

# Appendix 2a: Content

## Examples of Instructional Tasks for CONTENT

| Building Block | Pre-K | K-2 | Grades 3-5 |
|---|---|---|---|
| Integers | Students count up to 5 objects and match them to a numeral card. | Students listen to and tell each other simple story problems, such as "Tom has five toy cars. He gives two to his friend. How many toy cars does Tom have left?" They use physical objects, draw pictures, and then write a number sentence that corresponds with the story. | Students use expanded notation to decompose a three-digit number, multiply it by a single-digit number, and create a graphical representation to visualize this mathematical process. |
| Fractions | Students put together simple shape puzzles made of two or three large pieces to form a whole shape. | A teacher uses a 0–1 paper number line to introduce the idea that a whole can be divided into half and then into quarters; students fold and mark their paper number lines and color each quarter. | Students keep a diary that tracks their weekend time and then add the fractional hours to calculate the total number of hours they spend on their favorite activities. |
| Shapes | Students go on a simple shape hunt, finding objects with familiar shapes and naming the basic shape they see. | Students use different tools to measure dimensions of different objects in the classroom using standard units and comparing the relative dimensions of the objects. | Students use grid paper to place playground equipment on a playground, using different shapes for each type of equipment, with a limit of 100 square meters. They calculate the area of the footprint of each piece of equipment to ensure it fits into the available space, compare designs, and identify patterns. |
| Data | Students sort 6–8 images into two groups (e.g. cats and dogs) and count the number of images in each group. | Students design and administer surveys about class preferences (e.g., sports, food, etc.), create and present data visualizations, and discuss which visualizations were most helpful and why. | Students collect data about the weather each morning, represent the survey results graphically and analyze trends in different months. |

# Appendix 2b: Competencies

## Examples of Instructional Tasks for COMPETENCIES

| Building Block | Pre-K | K-2 | Grades 3-5 |
|---|---|---|---|
| Conceptual Understanding | Students use a small set of objects to make two equal groups, practicing matching one object to one object to build their understanding of one-to-one correspondence. | Students use counting tiles, simple drawings, and number bond diagrams to represent that 8+7 = (5+3) + 7 = 5+ (3+7) = 5+10 = 15 to build and demonstrate their Conceptual Understanding of equivalence, place value, and decomposition. | Students predict and record the outcomes of games (coin toss, dice roll, card draw, spinner) in a carnival to develop and demonstrate their Conceptual Understanding of certain, likely, unlikely and impossible outcomes as part of a foundation for learning probability. |
| Fact Fluency | Students use counters or dot cards to quickly recognize small quantities (1–3) shown in different arrangements without counting. | Students play Combinations of 10 Bingo. When the teacher calls a number, students find its "partner to 10" on their board (e.g., if the teacher says three, students look for a "seven" on their Bingo board and mark it out). | Students who know their fives multiplication combinations practice their sixes by rolling a 10-sided die. They reason: "five groups of [my number] equals _, so six groups of [my number] equals _," using known facts to build new ones. |
| Procedural Fluency | Students count a small group of up to five objects using one-to-one correspondence and practice recounting the same group after the objects are moved or rearranged to keep their counting accurate. | A student asked to subtract 25 from 44 notices that this problem is very similar to 45 – 25, which is 20 and adjusts their answer to 19 (since they added one to start the problem). Another student jumps up, thinking 25 to 35 is 10 and 35 to 44 is 9 more, correctly providing 19 as the answer. | Students adding 1/4 and 1/3 find a common denominator (12), convert each fraction to an equivalent fraction with the common denominator (3/12 and 4/12), add the numerators (7), and write the resulting fraction with the common denominator (7/12). |
| Application | Pairs of students play a ring toss game using two colors of rings. Each student tosses a small set of up to five rings. After tossing, they count how many rings of each color landed on the target and decide which color won the game. | Students make snack mixes containing different quantities of each ingredient and write recipe cards (e.g., one cup raisins, one cup pretzels, two cups M&Ms). | Students decide what to buy for a class party with a limited budget. They research the cost of supplies, decide what quantities to buy for each item, and ensure that they do not exceed their budget. |

# Appendix 2c: Ways of Thinking

## Examples of Instructional Tasks for WAYS OF THINKING

| Building Block | Pre-K | K-2 | Grades 3-5 |
|---|---|---|---|
| Symbolic Understanding | Students look at simple dot cards showing 1–3 dots and use their fingers or tally marks to show the same quantity. | Students match pictures, stories, and number equations with arrangements of base-ten blocks. | Students use fraction strips to compare and order fractions with different denominators by size relative to each other and to whole numbers. In reflection, they are asked to describe the meaning of the fraction bar. |
| Pattern Recognition | A teacher creates a repeating sound pattern using claps, stomps, and taps. Students listen to the pattern and try to recreate it, then create their own simple sound pattern for classmates to copy. | Students are given a sequence of whole numbers decreasing by a consistent rate with missing numbers; they must identify the missing numbers in a sequence and explain the pattern. | Students explore the pattern in a sequence of fractions with the same denominator (1/8, 2/8, 3/8) using fraction strips, graphing and words to describe the pattern. |
| Explaining | As a whole class, students count small sets of up to five objects in two different piles. Students are asked to say which pile has more or fewer and explain how they know. | A teacher uses the math language routine, "Stronger and Clearer Each Time," in which students explain how they got their answer to a math task to a partner, the partner asks a question (e.g., Why did you move these tiles?), and the explainer incorporates their answer to the question into a new, clearer explanation for a second partner. | After students are asked to solve the division problem 364 ÷ 4, the teacher asks students who used different strategies to each present their approach. |
| Sense-Making | Students look at two containers of different sizes and predict which one will hold more blocks. They explain their thinking, then test their prediction using real blocks. | A teacher uses a story to convey 22 – 7 = 15 and asks students to use two different strategies for depicting this equation, including base-ten blocks, number line, drawings and decomposition. Students then explain which strategy they prefer | Students analyze a graph about the increase in the price of a gallon of milk over the past 10 years, answering questions about how much the price has gone up, what years the price increased the most, and what impact this has had on their household's budget. |

# Appendix 2d: Motivators

## Examples of Instructional Tasks for MOTIVATORS

| Building Block | Pre-K | K-2 | Grades 3-5 |
|---|---|---|---|
| We Are All "Math People" | Students explore a small group of objects arranged on the carpet and share what they notice or wonder about them. Children offer different ideas and the teacher highlights that everyone sees math in their own way. | Each month, a teacher showcases a famous mathematician, including women and people of color. | A teacher uses a "pre-teaching" strategy, in which a student who has been struggling is tutored on the next day's lesson, so that student becomes the expert in the class on the new strategy. |
| Persistence Counts | Students try to build a block tower as tall as a toy figure. When the tower falls or leans, they try again in a different way and talk about what helped them keep going. | Students track their own progress toward fluency with addition facts up to 20 and celebrate when they reach a goal they have set. | Students use math journals to reflect on their problem-solving process, including challenges faced and how they overcame them, reinforcing the idea that effort leads to improvement. |
| Math Is Mind-Blowing! | Students pour water between containers of different shapes and sizes and notice how the same amount of water can look tall, short, wide, or narrow. They share what surprised them. | Students look at artistic works (e.g., M.C. Escher, Salvador Dali, Islamic art, Native American weavings), and create their own art piece in that style, and explain some aspect of their art using geometric terms. | Students are presented with various triangles and challenged to discover the relationships between the angles in the triangles, discovering that the sum of all angles in a triangle is always 180 degrees no matter the size or shape of the triangle. |
| Math Is Everywhere | Students help set the table by counting plates, cups, or napkins, noticing that they use math during everyday classroom routines. | Students ask family members for recipes and make a simple recipe in class. | Students create a class poll that includes both categorical and numerical questions, create graphs, and discuss mean, median, and mode. |

# Appendix 3
# Crosswalk: Common Core Standards for Mathematical Practice and Foundations of Numeracy

| | Common Core Standard for Mathematical Practice | Building Block |
|---|---|---|
| 1 | Make sense of problems and persevere in solving them. | Sense-Making + Persistence Counts |
| 2 | Reason abstractly and quantitatively. | Symbolic Understanding |
| 3 | Construct viable arguments and critique the reasoning of others. | Explaining |
| 4 | Model with mathematics. | Symbolic Understanding |
| 5 | Use appropriate tools strategically. | Sense-Making |
| 6 | Attend to precision. | Sense-Making |
| 7 | Look for and make use of structure. | Sense-Making |
| 8 | Look for and express regularity in repeated reasoning. | Pattern Recognition |

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